Dynamic Programming and Complexity Theory

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Dynamic programming

- Key idea: a problem can be solved combining the solution to a set of identically structured subproblems.
- The subproblems should be overlapping partially, or it's not dynamic programming it's divide et impera.
- Solution strategy:
  1. Spot the fact that the problem has an optimal substructure
  2. Locate the structure of a base-case solution
  3. Define a way to combine the solutions
- Example: cut bars with a given price list. Once cut rod for a given length is solved, reuse it

Top-Down Cut-rod-find-price(prices,n)

Input: prices: array of prices for the bars, the array index is the bar length, $n \in \mathbb{N}$: length of the bar
Output: $\langle \text{op}, \text{oc} \rangle$: best price and best cut points
Data: memopt: array of memoized best prices. initialized to all -1
memcuts: array of memoized best cut position lists

```
if memopt[n] ≠ −1 then
    return ⟨memopt[n], memcuts[n]⟩
if n = 0 then
    return ⟨0, 0⟩
op ← −1
oc ← []
for i = 1 to n do
    ⟨newprice, cutlist⟩ ← CUT-ROD-FIND-PRICE(prices, n − i)
    if op < prices[i] + newprice then
        op ← prices[i] + newprice
        oc ← CONCAT(i, cutlist)
memopt[n] ← op
memcuts[n] ← oc
return ⟨op, oc⟩
```

Bottom-Up Cut-rod-find-price($\texttt{prices}$,n)

Input: $\texttt{prices}$: array of prices for the bars, the array index is the bar length, $n \in \mathbb{N}$: length of the bar to cut
Output: $\langle \mathsf{op}, \mathsf{oc} \rangle$: best price and best cut points
Data: $\texttt{memopt}$: array of memoized best prices (init to -1), $\texttt{memcuts}$: array of memoized best first cuts

```
memopt[0] ← 0;
memcuts[0] ← 0;
if n = 0 then
    return ⟨0, 0⟩;
/* compute optima bottom up                                  */
for i ← 1 to n do
    memopt[i] ← −1;
    for j ← 1 to i do
        if memopt[i] < prices[j] + op[i − j] then
            memopt[i] ← prices[j] + op[i − j];
            memcuts[i] ← j;
oc ← [];
idx ← n;
while idx > 0 do
    PushBack(oc, memcuts[idx]);
    idx ← idx − memcuts[idx];
return ⟨op, oc⟩;
```

Greedy algorithms

In essence, when trying a solution to a sub-problem, pick the one that locally looks best.

- E.g., to find path with best weight in a tree, just look at the weight of your children to pick the direction

They usually do not provide the best solution, but they may have optimality guarantees

- Example 1: Dijkstra: picks the unvisited vertex with the shortest distance from the source, updates all connected nodes distances
- Example 2: Prim: build spanning tree starting from a vertex, adding the edge with lowest weight and its node. Repeat considering the edges outgoing from the current subgraph.

Complexity theory: purpose and subjects

- Complexity theory is used to classify problems according to how expensive in time/space is solving them
- We will deal with problems which are:
  - With both domain and range over the naturals $\mathbb{N}$
  - Corresponding to the computation of a *total computable function*: each problem has a finitely described algorithm solving it in finite time for any input
- A bit of problem taxonomy:
  - Search problem: given an input $x \in \mathbb{N}$ to a problem corresponding to the computation of $f(x)$, find $y = f(x), y \in \mathbb{N}$. Example: compute the square root of x.
  - Decision problem: given an input $x \in \mathbb{N}$ decide if x abides to some property $f(x) : \mathbb{N} \to \{\top, \bot\}$. Ex. Is x a perfect square?

Complexity classes

- Complexity of computing the solution to a problem as a function of the input length n in base $b > 1$
- Define the class (=set of problems) $\mathrm{DTIME}(f(n))$ as the ones for which a deterministic TM takes $f(n)$ moves to compute the solution
- $\mathrm{NTIME}(f(n))$ class: a nondeterministic TM takes $f(n)$ moves to compute the solution
- In general, relations between DTIME and NTIME are not well understood
  - Exception: $\mathrm{DTIME}(O(n)) \subset \mathrm{NTIME}(O(n))$
- Analog classes exist for space complexity $\mathrm{DSPACE}(f(n))$, $\mathrm{NSPACE}(f(n))$

Complexity classes

- Given $f(n)$, can always build a problem not in $\mathrm{DTIME}(f(n))$.
- $\forall k \geq 1$ there is a problem $\in \mathrm{DTIME}(n^k)$ and $\notin \mathrm{DTIME}(n^{k-1})$
- Some notable time complexity classes are:
  - $\mathrm{P} = \bigcup_{i \geq 1} \mathrm{DTIME}(n^i), i \in \mathbb{N}$: "practically treatable" for any n
  - $\mathrm{NP} = \bigcup_{i \geq 1} \mathrm{NTIME}(n^i), i \in \mathbb{N}$
  - $\mathrm{EXP} = \bigcup_{i \geq 1} \mathrm{DTIME}(2^{n^i}), i \in \mathbb{N}$
  - $\mathrm{PSPACE} = \bigcup_{i \geq 1} \mathrm{DSPACE}(n^i), i \in \mathbb{N}$: NOT practically tractable in general, $\mathrm{NP} \subseteq \mathrm{PSPACE}$
- $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXP}$, but $\mathrm{P} \subsetneq \mathrm{EXP}$
- Open questions: $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$, $\mathrm{P} \stackrel{?}{=} \mathrm{PSPACE}$. Likely answer: No.

NP - An alternate definition

- It is possible to define a problem to be in NP in two ways
  1. There is a nondeterministic TM which computes the solution to it in polynomial time
  2. There is a deterministic TM which verifies that a solution for the problem is an actual solution in polynomial time
     - For decision problems: there is a deterministic TM which, given an element which makes the ND-TM accept, tests that is actually one of the elements which should make the ND-TM accept
- Example: Exiting a binary branching labyrinth without a map:
  1. A nondeterministic TM will find the poly-length exiting path, if any, taking all the branches in parallel
  2. A deterministic TM, given the path, will verify that it actually exits from the labyrinth through walking through it

NP and complement classes

- P and NP defined on decision problems, for simplicity.
- A decision problem on the naturals $=$ test if a natural belongs to some defined set $=$ test if a string belongs to a language
- What about testing if the integer does not belong to a set?
  - If recognizing if it belongs to the set $\in \mathrm{P}$ the problem is still $\in \mathrm{P}$: swap accepting/rejecting states in the recognizing TM.
  - If recognizing if it belongs to the set $\in \mathrm{NP}$, the problem is $\in \mathrm{coNP}$: the class of problems for which (deciding if an elements belongs to) the complement of a given set is in nondet-poly time.
  - The "swap accepting/rejecting" does not work anymore: a TM terminates on a single accepting state, or when all paths reject
    - If I swap states, I'll only check one of the old rejecting paths
  - $\mathrm{NP} = \mathrm{coNP}$? Open question; Highly likely answer: no.

Computational reductions

- A need tool to compare the complexity of two problems: computational (aka Turing) reduction
- Given two problems A and B, A reduces to B if given an oracle for B I can solve A. Thus
  - A cannot be harder to solve than B
  - B can be harder to solve than A
  - Therefore, $A \leq^T B$ (where T reminds it's a Turing reduction)
- If $A \leq^T B$ and I make only a poly number of calls to the oracle for B when solving A, plus extra poly-time computations
  - A reduces polynomially (or, Cook-reduces) to B
  - $A \leq_p^T B$, since solving B also solves A with extra poly effort

CLASS-hardness and CLASS-completeness

- Given a complexity class CLASS and a generic problem B, B is CLASS-hard iff:
    - $\forall A \in \mathrm{CLASS}, A \leq_p^T B$
- In other words, solving a CLASS-hard problem solves with extra poly effort all the problems in CLASS.
- Given a complexity class CLASS and a problem B, B is CLASS-complete iff:
    - $B \in \mathrm{CLASS}$ and B is CLASS-hard
- CLASS-complete problems are the “computationally hardest” within a class

A taxonomy in NP – Assuming $\mathrm{P} \neq \mathrm{NP}$

- NP-Hard problem: any problem such that I can poly reduce to it any problem in NP. Note, there may be non-decision problems in this set.
- NP-Complete class: problems such that all the problems in NP can be poly reduced to any of them. Conventially formulated as decision problems only
  - 3-SAT,Graph coloring, Subgraph isomorphism, decoding random linear codes, syndrome decoding
- NP-intermediate: essentially the complement of NPC to NP$\setminus$P. Some problems here have sub-exponential solutions
  - Factoring $\in \mathrm{DTIME}(\mathcal{O}(e^{(c+o(1))n^{\frac{1}{3}}\log(n)^{\frac{2}{3}}}))$, Discrete Logarithms
  - Graph isomorphism (proven in 2015, disproved in '17, reproven in '17 later on, should be $\mathcal{O}(2^{(\log(n)^3)})$, this time, for real)

Quantum Computing Model

- A transition of a classical TM computes a constant time operation on a finite set of symbols from a finite alphabet
- An abstract quantum computing machine is a machine applying unitary transformations (gates) assumed to be constant time to a finite set of qubits
- Complexity evaluated as a function of the (classical) input length n:
  - Time: evaluated as either the number of sequential gates (depth) or the total number of gates ($\mathcal{O}(\text{depth}\times\text{qubits})$)
  - Space: evaluated as number of involved qubits
- A measurement of the result may not yield the same classical value if the computation is repeated $\rightarrow$ probabilistic computation model

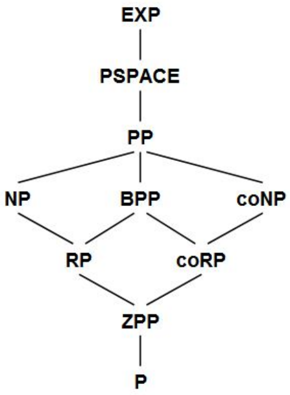
A primer on probabilistic computation

- What is a randomized algorithm? We may randomize:
  - running time: Algorithm A runs in probabilistic poly time, i.e., with a certain probability it terminates in poly time
  - correctness: regardless of the running time, the algorithm returns the correct answer with probability p
- We can solve problems, in expectation, if:
  - The running time is either deterministically or probabilistically polynomial, with $Pr(\text{poly time}) \gg \frac{1}{2}$ink
  - The algorithm provides a corrects solution with a satisfactory probability, that is, either :
    - $p \approx 1$
    - p is large enough to allow us to query the algo poly times and do majority voting. $p = \frac{1}{2} + \frac{1}{2^n}$ would need exp. no. calls

Probabilistic computation classes (for TMs)

- PP class: the problem is solved in $\mathrm{DTIME}(poly(n))$ by an algorithm outputting the correct answer with $\Pr > \frac{1}{2}$
  - Not necessarily tractable: $\frac{1}{2} + \frac{1}{2^n} > \frac{1}{2}$
- BPP class: the problem is solved in $\mathrm{DTIME}(poly(n))$ by an algorithm outputting the correct answer with $\Pr > \frac{1}{2} + k$
  - Tractable, for reasonable values of constant k, Typ. $\Pr = \frac{2}{3}$
- RP class: the problem is solved in $\mathrm{DTIME}(poly(n))$ by an algorithm outputting accept with $\Pr > \frac{1}{2} + k$,if the solution is accept, but *never* accepting when it has to reject
- ZPP class: the problem is solved in $\mathrm{DTIME}(poly(n))$ by an algorithm which: gives a correct answer with $\Pr = \frac{1}{2}$, or answers "I don't know" with $\Pr = \frac{1}{2}$.
  - Equivalent to an algo running on *expected* poly time, always giving correct answers (not straightforward)

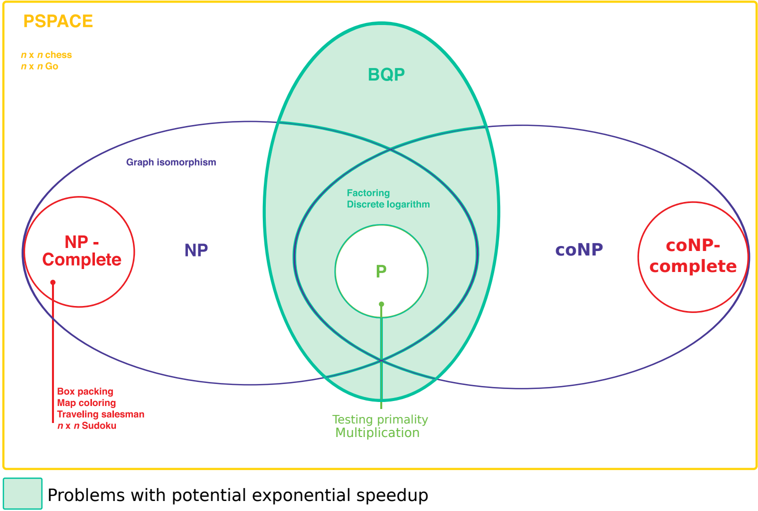
Probabilistic computation classes (for TMs)



Probabilistic computation classes (for quantum computers)

- Finally, we can define what is tractable by a QC
  - We consider the former abstract quantum machine model and define complexity as a function of the classical input length n
- BQP class: the problem is solved in $\mathrm{DTIME}(poly(n))$ by a quantum machine which leaves a final state on which a measure yields the classical correct answer with $\Pr > \frac{1}{2} + k$
- Functionally analogous to BPP, but on a quantum computer
- We know $\mathrm{BPP} \subseteq \mathrm{BQP}$: a QC emulates a classical probabilistic TM on poly-time algs, with poly-time overhead
- No sense in defining QP: cannot have a deterministic QC!

Relations between BQP and the other classes



Which speedups can we achieve?

- Anything lying in $\mathrm{NP} \setminus \mathrm{P}$, and in BQP, or in $\mathrm{coNP} \setminus \mathrm{P}$, and in BQP $\rightarrow$ likely to gain exponential speedup
  - We have no general det. poly algorithm for $A \in \mathrm{NP} \setminus \mathrm{P}$
- Anything in P $\rightarrow$ likely not worth it: already poly time
  - Usually, solving problems in P on a QC is slower (due probabilistic computation and reversibility)
- Outside BQP, inside PSPACE $\rightarrow$ no exponential speedup, but subexponential gains are possible
  - E.g. complexity goes down from $\mathcal{O}(2^n)$ to $\mathcal{O}(2^{\frac{n}{2}})$
- Strong belief: $\mathrm{NP\text{-}complete} \cap \mathrm{BQP} = \varnothing$
  - A quantum computer is not a nondeterministic TM!
  - 3SAT, SubGI are not getting an exponential speedup